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Simulation of Brown–Resnick processes

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Abstract Brown–Resnick processes form a flexible class of stationary max-stable processes based on Gaussian random fields. With regard to applications, fast and accurate simulation of these processes is an important issue. In fact, Brown–Resnick processes that are generated by a dissipative flow do not allow for good finite approximations using the definition of the processes. On large intervals we get either huge approximation errors or very long operating times. Looking for solutions of this problem, we give different representations of the Brown–Resnick processes—including random shifting and a mixed moving maxima representation—and derive various kinds of finite approximations that can be used for simulation purposes. Furthermore, error bounds are calculated in the case of the original process by Brown and Resnick (J Appl Probab 14(4):732–739, 1977). For a one-parametric class of Brown–Resnick processes based on the fractional Brownian motion we perform a simulation study and compare the results of the different methods concerning their approximation quality. The presented simulation techniques turn out to provide remarkable improvements.

Keywords Error estimate · Extremes · Gaussian process · Max-stable process · Poisson point process

AMS 2000 Subject Classifications 60G70 · 60G10 · 68U20

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1 Introduction

Stochastically continuous max-stable processes have been entirely characterized by de Haan (1984). Based on this approach many models for stationary max-stable processes have been considered. For instance, Smith (1990) introduced "rainfall-storm" models like the Gaussian and t extreme value processes. Further models are given in Schlather (2002) and de Haan and Pereira (2006), see also de Haan and Ferreira (2006).

We will focus on a class of stationary max-stable processes that has been introduced by Brown and Resnick (1977), further investigated by Falk et al. (2004), and generalized by Kabluchko et al. (2009). This class is notable, as a subclass also occurs as the limit of maxima of independent copies of stationary and appropriately scaled Gaussian random fields (Kabluchko et al. 2009) and, in a modified form, as the limit of empirical distribution functions (Kabluchko 2009a). Ergodic properties of the processes are studied by Kabluchko (2009b) and Wang and Stoev (2009) who discuss the decomposition into conservative and dissipative components; ergodicity and mixing properties are investigated by Kabluchko and Schlather (2009). Finally, Buishand et al. (2008) and de Haan and Zhou (2008) used Brown–Resnick processes for modelling spatial rainfall.

This paper is organized as follows: in Section 2 we introduce the class of Brown–Resnick processes. Equivalent representations of these processes based on random shifts are presented in Section 3. Section 4 deals with those Brown–Resnick processes which are generated by a dissipative flow and discusses further representations for them. All these different representations lead to different kinds of finite approximations introduced in Section 5. Error estimates for these approximations are given in Section 6 for the original process of Brown and Resnick (1977). In Section 7, we compare the quality of different approximations by means of a simulation study.

Here, we restrict ourselves to max-stable processes with Gumbel margins. Fréchet and Weibull margins are obtained by marginal transformation.

2 Basics

A stochastic process $\{Z(t),\ t \in \mathbb{R}^d\}$ with Gumbel margins is called *max-stable* if the processes $\{\max_{i=1,\dots,n} Z_i(t) - \log n,\ t \in \mathbb{R}^d\}$ and $\{Z(t),\ t \in \mathbb{R}^d\}$ have the same law for any $n \in \mathbb{N}$, where $\{Z_i(t),\ t \in \mathbb{R}^d\}$ are independent copies of $\{Z(t),\ t \in \mathbb{R}^d\}$.

Let $\{W(t),\ t \in \mathbb{R}^d\}$ be a Gaussian process with stationary increments, that is, the law of $\{W(t+h) - W(h),\ t \in \mathbb{R}^d\}$ does not depend on the choice of $h \in \mathbb{R}^d$. For any second-order process $W(\cdot)$ with stationary increments the *variogram* $\gamma(\cdot)$ (see Chilès and Delfiner 1999) is defined by

$$\gamma(h) = \mathbb{E}(W(h) - W(0))^2, \quad h \in \mathbb{R}^d.$$

If $W(0) = 0$, then $\sigma^2(t) = \gamma(t)$, where $\sigma^2(t)$ denotes the variance $\mathrm{Var}(W(t))$. We recall the construction of stationary max-stable processes, which has been introduced by Brown and Resnick (1977) in case of $W(\cdot)$ being a Brownian motion.

Theorem 1 (Kabluchko et al. 2009) *Let* $\{W_i(t),\ t \in \mathbb{R}^d\}$, $i \in \mathbb{N}$, *be independent copies of a Gaussian random field* $\{W(t),\ t \in \mathbb{R}^d\}$ *with zero mean and variance* $\sigma^2(\cdot)$. *Independently, let* $\sum_{i\in\mathbb{N}} \delta_{X_i}$ *be a Poisson point process on* $\mathbb{R}$ *with intensity* $\exp(-x)\mathrm{d}x$.

1. *The process* $\{Z(t),\ t \in \mathbb{R}^d\}$, *defined by*

$$Z(t) = \max_{i\in\mathbb{N}} \left(X_i + W_i(t) - \frac{\sigma^2(t)}{2} \right),$$

   is a max-stable process with standard Gumbel margins.
2. *If, additionally,* $W(\cdot)$ *has stationary increments, then the process* $Z(\cdot)$ *is stationary and its law only depends on the variogram* $\gamma(\cdot)$ *of* $W(\cdot)$. *The process* $Z(\cdot)$ *is called Brown–Resnick process associated to the variogram* $\gamma(\cdot)$.
3. *Moreover, under the assumptions above,* $\sum_{i\in\mathbb{N}} \delta_{X_i+W_i(\cdot)-\sigma^2(\cdot)/2}$ *is a translation invariant Poisson point process on* $\mathbb{R}^{\mathbb{R}^d}$.

3 Random shifts

Figure 1 shows that a finite approximation of the Brown–Resnick process based on the definition turns out to appear non-stationary on large intervals if the equation

$$\lim_{||t||\to\infty} \left(W(t) - \frac{\sigma^2(t)}{2} \right) = -\infty \quad \mathbb{P} - a.s. \tag{1}$$

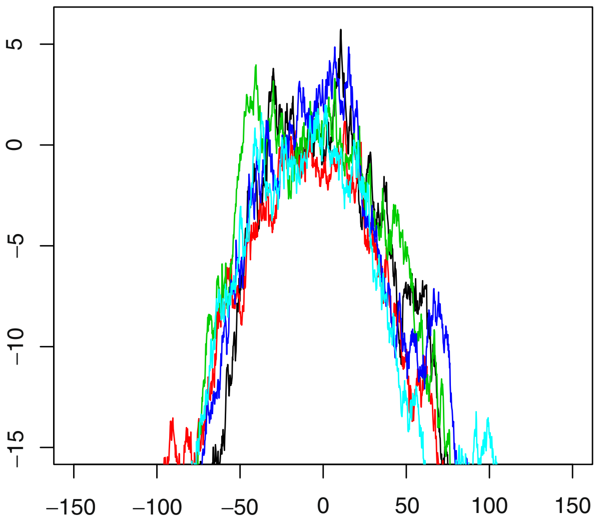

Fig. 1 Five finite approximations of the original Brown–Resnick process, each based on the largest 1,000 values of the underlying Poisson point process.

holds. Therefore, we seek equivalent representations of Brown–Resnick processes that avoid this drawback. A first possibility is to include some "random shifting" into the construction.

Theorem 2 *Let $W_i(\cdot)$, $i \in \mathbb{N}$, be independent copies of a Gaussian random field $\{W(t),\ t \in \mathbb{R}^d\}$ with zero mean, stationary increments and variance $\sigma^2(\cdot)$ and let Q be a probability measure on $\mathbb{R}^d$. Independently of $W_i(\cdot)$, let $\sum_{i\in\mathbb{N}} \delta_{(X_i,H_i)}$ be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity measure* $\exp(-x)\mathrm{d}x \times Q(\mathrm{d}h)$. *Then,*

$$\tilde{Z}(t) = \max_{i\in\mathbb{N}} \left(X_i + W_i(t - H_i) - \frac{\sigma^2(t - H_i)}{2} \right), \quad t \in \mathbb{R}^d,$$

is a Brown–Resnick process associated to the variogram $\gamma(\cdot)$, i.e. $\tilde{Z} \stackrel{d}{=} Z$.

Proof Let $t_1, \ldots, t_m \in \mathbb{R}^d,\ y_1, \ldots y_m \in \mathbb{R}$ and $m \in \mathbb{N}$, be arbitrary and $\mathbb{P}_{t_1,\ldots,t_m}$ be the law of the random vector $(W(t_1), \ldots, W(t_m))$.

Then, $\Phi = \sum_{i\in\mathbb{N}} \delta_{(X_i,H_i,W_i)}$ is a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathbb{R}^d}$ with intensity measure $\exp(-x)\mathrm{d}x \times Q(\mathrm{d}h) \times \mathbb{P}_W(dw)$ and

$$\tilde{Z}(t_1) \le y_1, \ldots, \tilde{Z}(t_m) \le y_m \iff \Phi\left(\left\{(x, h, w) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{\mathbb{R}^d} : \right.\right.$$
$$\left.\left. x > \min_{k=1,\ldots,m} \left(y_k - w(t_k - h) + \frac{\sigma^2(t_k - h)}{2} \right) \right\}\right) = 0.$$

Thus, we obtain

$$\begin{aligned}
&- \log\left(\mathbb{P}(\tilde{Z}(t_1) \le y_1, \ldots, \tilde{Z}(t_m) \le y_m)\right) \\
&= \int_{\mathbb{R}^d} \int_{\mathbb{R}^m} \int_{\min_{k=1,\ldots,m}\left(y_k - w_k + \frac{\sigma^2(t_k-h)}{2}\right)}^{\infty} \exp(-x)\mathrm{d}x \\
&\quad \times \mathbb{P}_{t_1-h,\ldots,t_m-h}(\mathrm{d}w_1, \ldots, \mathrm{d}w_m)\ Q(\mathrm{d}h) \\
&= \int_{\mathbb{R}^d} - \log(\mathbb{P}(Z(t_1 - h) \le y_1, \ldots, Z(t_m - h) \le y_m))\ Q(\mathrm{d}h).
\end{aligned}$$

Due to the stationarity of $Z(\cdot)$ the right hand side equals

$$- \log\left(\mathbb{P}(Z(t_1) \le y_1, \ldots, Z(t_m) \le y_m)\right).$$

□

This theorem can be used for representing Brown–Resnick processes in many different ways. Here, we give two corollaries as applications.

Corollary 3 *Let $W(\cdot)$ be as in Theorem* 1, $W_i^{(j)} \sim_{i.i.d.} W$, $i \in \mathbb{N}$, $j = 1, \ldots, n$. *Independently of* $W_i^{(j)}$, $i \in \mathbb{N}$, *let* $\Pi^{(j)} = \sum \delta_{X_i^{(j)}}$, $j = 1, \ldots, n$, *be independent Poisson point processes on* $\mathbb{R}$ *with intensity measure* $n^{-1}\exp(-x)\mathrm{d}x$ *and* $h_1, \ldots, h_n \in \mathbb{R}^d$. *Then,*

$$Z_1(t) = \max_{j=1,\ldots,n} \max_{i\in\mathbb{N}} \left(X_i^{(j)} + W_i^{(j)}(t-h_j) - \frac{\sigma^2(t-h_j)}{2} \right), \quad t \in \mathbb{R}^d,$$

is a Brown–Resnick process associated to the variogram $\gamma(\cdot)$, i.e. $Z_1 \stackrel{d}{=} Z$.

Proof Note that the superposition $\sum_{j=1}^n \sum_{i\in\mathbb{N}} \delta_{(X_i^{(j)}, h_j)}$ is a Poisson point process on $\mathbb{R} \times \mathbb{R}^d$ with intensity measure $\exp(-x)\mathrm{d}x \times (\frac{1}{n}\sum_{j=1}^n \delta_{h_j})$, and apply Theorem 2. □

Corollary 4 *Let $W_i(\cdot)$ be as in Theorem* 1, *and $I \subset \mathbb{R}^d$ a finite cuboid. Independently of W_i let $\Pi = \sum \delta_{(X_i, H_i)}$ be a Poisson point process on $\mathbb{R} \times I$ with intensity measure* $\exp(-x)\mathrm{d}x \times |I|^{-1}\mathrm{d}h$. *Then,* $Z \stackrel{d}{=} Z_2$, *where*

$$Z_2(t) = \max_{i\in\mathbb{N}} \left(X_i + W_i(t-H_i) - \frac{\sigma^2(t-H_i)}{2} \right), \quad t \in \mathbb{R}^d.$$

Proof With $Q(\mathrm{d}h) = |I|^{-1}\mathbf{1}_{h\in I}\mathrm{d}h$ the assertion follows from Theorem 2. □

4 Mixed moving maxima representation

The notion of max-stable processes generated by non-singular flows has been introduced by de Haan and Pickands (1986); further results on the representations of max-stable processes have been obtained in Kabluchko (2009b) and Wang and Stoev (2009) by transferring some work of Rosiński (1995) on $S\alpha S$-processes.

Kabluchko et al. (2009, Theorem 14) showed that a Brown–Resnick process is generated by a dissipative flow if Eq. 1 holds. In the case $d = 1$ condition (1) is satisfied if $\liminf_{t\to\infty} \gamma(t)/\log t > 8$.

Using the stationarity criterion from the third part of Theorem 1, we will be able to provide equivalent representations of Brown–Resnick processes given by the following theorems.

Theorem 5 *Let* $W_i^{(j)}$, $i \in \mathbb{N}$, $j \in \mathbb{Z}^d$, *be independent copies of a Gaussian random field $W(\cdot)$ with continuous sample paths, stationary increments, zero mean, variance $\sigma^2(\cdot)$ and variogram $\gamma(\cdot)$ on $\mathbb{R}^d$. Furthermore, let*

$$T_i^{(j)} = \inf\left(\mathrm{argsup}_{t\in\mathbb{R}^d} \left(W_i^{(j)}(t) - \frac{\sigma^2(t)}{2} \right) \right)$$

where the "inf" is understood in the lexicographic sense if $d > 1$. *We assume Eq.* 1, *so that* $T_i^{(j)}$ *is well-defined a.s.*

Independently of $W_i^{(j)}$, *let* $\Pi^{(j)} = \sum_{i\in\mathbb{N}} \delta_{X_i^{(j)}}$, $j \in \mathbb{Z}^d$, *be independent Poisson point processes on* $\mathbb{R}$ *with intensity measure* $m^{-d}\exp(-x)\mathrm{d}x$ *for some* $m \in \mathbb{N}$. *Furthermore, let* $p > 0$. *Then,* $Z_3 \stackrel{d}{=} Z$ *where*

$$Z_3(t) = \max_{j\in\mathbb{Z}^d} \max_{\substack{i\in\mathbb{N} \\ T_i^{(j)} \in (-\frac{m}{2}p, \frac{m}{2}p]^d}} \left(X_i^{(j)} + W_i^{(j)}(t-pj) - \frac{\sigma^2(t-pj)}{2} \right), \quad t \in \mathbb{R}^d,$$

is a Brown–Resnick process associated to the variogram $\gamma(\cdot)$.

Proof Let $\mathscr{C}$ be the σ-algebra on $C(\mathbb{R}^d)$ generated by the sets

$$C_{t_1,\ldots,t_m}(B) = \{f \in C(\mathbb{R}^d) : (f(t_1), \ldots, f(t_m)) \in B\}$$

with $t_1, \ldots, t_m \in \mathbb{R}^d$, $m \in \mathbb{N}$ and $B \in \mathscr{B}^m$, where $\mathscr{B}^m$ is the Borel-σ-algebra of $\mathbb{R}^m$. Endowed with the topology of uniform convergence on compact sets, $C(\mathbb{R}^d)$ becomes a Polish space and the σ-algebra $\mathscr{C}$ coincides with the Borel-σ-algebra generated by this topology.

We define $\xi_i^{(j)}(t) = W_i^{(j)}(t) - \sigma^2(t)/2$. Because of condition (1) each $T_i^{(j)}$ is finite $\mathbb{P}$-a.s. and $M_i^{(j)} = \sup_{t\in\mathbb{R}^d}(X_i^{(j)} + \xi_i^{(j)}(t))$ is well-defined. The mapping

$$\Phi : C(\mathbb{R}^d) \to \mathbb{R}^d \times C(\mathbb{R}^d), \quad X_i^{(j)} + \xi_i^{(j)}(\cdot) \mapsto \left(T_i^{(j)}, X_i^{(j)} + \xi_i^{(j)}(\cdot)\right),$$

is measurable since $\sup_{x\in\mathbb{R}^d} \xi_i^{(j)}(x) = \sup_{x\in\mathbb{Q}^d} \xi^{(j)}(x)$ and $T_i^{(j)}$ is the first root of $\xi_i^{(j)} - \sup(\xi_i^{(j)})$. Therefore, the mapping theorem for Poisson point processes (Kingman 1993) yields that $\sum_{i\in\mathbb{N}} \delta_{(T_i^{(j)}, X_i^{(j)} + \xi_i^{(j)}(\cdot))}$ is a Poisson point process with intensity measure

$$\Psi(A) = \int_{\mathbb{R}} \frac{1}{m^d} \exp(-x)\mathbb{P}_W\left(x + \xi \in \Phi^{-1}(A)\right)\mathrm{d}x, \quad A \in \mathscr{B}^d \times \mathscr{C},$$

where $\mathbb{P}_W$ is the law of the process $W(\cdot)$.

Now we define $U_t : C(\mathbb{R}^d) \to C(\mathbb{R}^d)$, $f(\cdot) \mapsto f(\cdot - t)$ and $V_t : \mathbb{R}^d \times C(\mathbb{R}^d) \to \mathbb{R}^d \times C(\mathbb{R}^d)$, $(x, f(\cdot)) \mapsto (x+t, f(\cdot - t))$ as translations by $t \in \mathbb{R}^d$. Then we obtain

$$\begin{aligned}(\Phi \circ U_t)\left(X_i^{(j)} + \xi_i^{(j)}(\cdot)\right) &= \left(T_i^{(j)} + t, X_i^{(j)} + \xi_i^{(j)}(\cdot - t)\right) \\ &= (V_t \circ \Phi)\left(X_i^{(j)} + \xi_i^{(j)}(\cdot)\right).\end{aligned}$$

The intensity measure of the Poisson point process $\sum \delta_{X_i^{(j)} + \xi_i^{(j)}(\cdot)}$ is translation invariant (with respect to U_t) by Theorem 1. Because of the fact that Φ commutes with the translation operators, Ψ is translation invariant (with respect to V_t), as well.

Thus, for any $j \in \mathbb{Z}^d$, we obtain

$$\max_{\substack{i\in\mathbb{N}\\ T_i^{(j)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d}} \left(X_i^{(j)} + \xi_i^{(j)}(\cdot - pj)\right) \stackrel{d}{=} \max_{\substack{i\in\mathbb{N}\\ T_i^{(j)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d+pj}} \left(X_i^{(j)} + \xi_i^{(j)}(\cdot)\right). \tag{2}$$

Now we consider each side of Eq. 2 separately. For different $j \in \mathbb{Z}^d$ we get stochastically independent processes. This yields

$$\begin{aligned} Z_3(\cdot) &= \max_{j\in\mathbb{Z}^d} \max_{\substack{i\in\mathbb{N}\\ T_i^{(j)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d}} \left(X_i^{(j)} + \xi_i^{(j)}(\cdot - pj)\right) \\ &\stackrel{d}{=} \max_{j\in\mathbb{Z}^d} \max_{\substack{i\in\mathbb{N}\\ T_i^{(j)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d+pj}} \left(X_i^{(j)} + \xi_i^{(j)}(\cdot)\right). \end{aligned}$$

Furthermore, by replacing $T_i^{(j)}$, $\xi_i^{(j)}$, and $X_i^{(j)}$ by $T_i^{(j \bmod m)}$, $\xi_i^{(j \bmod m)}$, and $X_i^{(j \bmod m)}$, respectively, we obtain

$$\begin{aligned} &\max_{\substack{i\in\mathbb{N}\\ T_i^{(j)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d+pj}} \left(X_i^{(j)} + \xi_i^{(j)}(\cdot)\right) \\ &\qquad \stackrel{d}{=} \max_{\substack{i\in\mathbb{N}\\ T_i^{(j \bmod m)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d+pj}} \left(X_i^{(j \bmod m)} + \xi_i^{(j \bmod m)}(\cdot)\right) \end{aligned}$$

where "mod" is understood as a componentwise operation.

For $j_1 \equiv j_2 \bmod m$, $j_1 \neq j_2$ we have

$$((-mp/2, mp/2]^d + pj_1) \cap ((-mp/2, mp/2]^d + pj_2) = \emptyset,$$

which guarantees that processes $\xi_i^{(j_1 \bmod m)}$ with $T_i^{(j_1 \bmod m)} \in (-mp/2, mp/2]^d + pj_1$ and $\xi_i^{(j_2 \bmod m)}$ with $T_i^{(j_2 \bmod m)} \in (-mp/2, mp/2]^d + pj_2$ are independent. By these considerations we get

$$\begin{aligned} Z_3(\cdot) &\stackrel{d}{=} \max_{j\in\mathbb{Z}^d} \max_{\substack{i\in\mathbb{N}\\ T_i^{(j \bmod m)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d+pj}} \left(X_i^{(j \bmod m)} + \xi_i^{(j \bmod m)}(\cdot)\right) \\ &\stackrel{d}{=} \max_{k\in\{0,\ldots,m-1\}^d} \max_{\substack{j\in\mathbb{Z}^d\\ j \bmod m\equiv k}} \max_{\substack{i\in\mathbb{N}\\ T_i^{(k)}\in(-\frac{m}{2}p,\frac{m}{2}p]^d+pj}} \left(X_i^{(k)} + \xi_i^{(k)}(\cdot)\right) \\ &= \max_{k\in\{0,\ldots m-1\}^d} \max_{i\in\mathbb{N}} \left(X_i^{(k)} + \xi_i^{(k)}(\cdot)\right) \stackrel{d}{=} Z(\cdot). \end{aligned}$$

The last step is based on the fact that $\sum_{k\in\{0,\dots,m-1\}^d}\sum_{i\in\mathbb{N}}\delta_{X_i^{(k)}}$ is a Poisson point process with intensity measure $\sum_{k\in\{0,\dots,m-1\}^d} m^{-d}\exp(-x)\mathrm{d}x = \exp(-x)\mathrm{d}x$. □

By Kabluchko (2009b) condition (1) holds only if $Z(\cdot)$ has a mixed moving maxima representation. In order to construct such a representation we repeat results from the proof of Theorem 14 in Kabluchko et al. (2009).

Theorem 6 *Let* $\{W_i(t),\ t\in\mathbb{R}^d\}$, $i\in\mathbb{N}$, *be independent copies of a Gaussian random field* $\{W(t),\ t\in\mathbb{R}^d\}$ *with continuous sample paths, stationary increments, zero mean, variance* $\sigma^2(\cdot)$ *and variogram* $\gamma(\cdot)$ *on* $\mathbb{R}^d$. *We assume that condition* (1) *is satisfied. Furthermore, let* $T_i = \inf\left(\mathrm{argsup}_{t\in\mathbb{R}^d}\left(W_i(t)-\frac{\sigma^2(t)}{2}\right)\right)$, $M_i=\sup_{t\in\mathbb{R}^d}\left(W_i(t)-\frac{\sigma^2(t)}{2}\right)$ *and* $F_i(\cdot)=W_i(\cdot+T_i)-\frac{\sigma^2(\cdot+T_i)}{2}-M_i$.

Independently of W_i, *let* $\sum_{i\in\mathbb{N}}\delta_{X_i}$ *be a Poisson point process with intensity measure* $\exp(-x)\mathrm{d}x$. *Then, the random measure* $\sum_{i\in\mathbb{N}}\delta_{(T_i,X_i+M_i,F_i)}$ *defines a Poisson point process on* $\mathbb{R}^d\times\mathbb{R}\times C(\mathbb{R}^d)$ *with intensity measure* $\lambda^*\mathrm{d}t\times e^{-y}\mathrm{d}y\times\tilde{Q}(\mathrm{d}F)$ *for some* $\lambda^*>0$ *and a probability measure* $\tilde{Q}$ *on* $C(\mathbb{R}^d)$. *Furthermore, let* $\sum_{i\in\mathbb{N}}\delta_{(S_i,U_i)}$ *be a Poisson point process on* $\mathbb{R}^d\times\mathbb{R}$ *with intensity measure* $\lambda^*\mathrm{d}te^{-u}\mathrm{d}u$ *and* $\tilde{F}_i\sim_{i.i.d.}\tilde{Q}$. *Then, we have* $Z_4\stackrel{d}{=}Z$ *for*

$$Z_4(t)=\max_{i\in\mathbb{N}}\left(U_i+\tilde{F}_i(t-S_i)\right),\quad t\in\mathbb{R}^d.$$

Proof The first part is shown in the proof of Theorem 14 in Kabluchko et al. (2009).

For the second part note that $\sum_{i\in\mathbb{N}}\delta_{(T_i,X_i+M_i,F_i)}$ and $\sum_{i\in\mathbb{N}}\delta_{(S_i,U_i,\tilde{F}_i)}$ are Poisson point processes on $\mathbb{R}^d\times\mathbb{R}\times C(\mathbb{R}^d)$ with the same intensity measure. Furthermore, we have $Z(\cdot)=\max_{i\in\mathbb{N}}\Gamma(T_i,X_i+M_i,F_i)$ and $Z_4(\cdot)=\max_{i\in\mathbb{N}}\Gamma(S_i,U_i,\tilde{F}_i)$ with the transformation

$$\Gamma:\mathbb{R}^d\times\mathbb{R}\times C(\mathbb{R}^d)\to C(\mathbb{R}^d),\ (u,s,f)\mapsto u+f(\cdot-s).$$

□

Remark 7 A similar result holds if we consider all the processes from Theorem 6 restricted to $p\mathbb{Z}^d$, $p>0$, instead of $\mathbb{R}^d$.

Then, for

$$T_i^{(p)}=\inf\left(\mathrm{argsup}_{t\in p\mathbb{Z}^d}\left(W_i(t)-\frac{\sigma^2(t)}{2}\right)\right),$$

$$M_i^{(p)}=\sup_{t\in p\mathbb{Z}^d}\left(W_i(t)-\frac{\sigma^2(t)}{2}\right),$$

and $$F_i^{(p)}(\cdot)=W_i\left(\cdot+T_i^{(p)}\right)-\frac{\sigma^2\left(\cdot+T_i^{(p)}\right)}{2}-M_i^{(p)},\quad t\in p\mathbb{Z}^d,$$

the random measure $\sum_{i\in\mathbb{N}} \delta_{(T_i^{(p)}, X_i+M_i^{(p)}, F_i^{(p)})}$ defines a Poisson point process on $p\mathbb{Z}^d \times \mathbb{R} \times \mathbb{R}^{p\mathbb{Z}^d}$ with intensity measure $\lambda^{(p)} p^d \delta_t \times e^{-u}\mathrm{d}u \times \tilde{Q}^{(p)}(\mathrm{d}F)$ for some $\lambda^{(p)} > 0$ and some probability measure $\tilde{Q}^{(p)}$. An equivalent representation Z_4 of Z can be given analogously to Theorem 6. Even more easily, all the other results from Sections 2, 3 and 4 can be transferred to processes on a lattice.

For approximating Z via representation Z_4 the law $\tilde{Q}$ is needed explicitly. Note that, in general $\tilde{Q}$ is not the law of $W(\cdot + T) - \sigma^2(\cdot + T) - M$ (and $\tilde{Q}^{(p)}$ is not the law of $W(\cdot + T^{(p)}) - \sigma^2(\cdot + T^{(p)}) - M^{(p)}$). If we assume $W(0) = 0$ and restrict ourselves to processes on a lattice $p\mathbb{Z}^d$, we get the following result.

Theorem 8 *Let $W(\cdot)$ be as in Theorem 6 and*

$$T^{(p)} = \inf\left(\mathrm{argsup}_{t\in p\mathbb{Z}^d}\left(W(t) - \frac{\sigma^2(t)}{2}\right)\right).$$

Furthermore, assume $W(0) = 0$. Then, $Q^{(p)}$ is the law of

$$W(\cdot) - \frac{\sigma^2(\cdot)}{2} \mid T^{(p)} = 0.$$

Proof Let $A \in \mathscr{B}(\mathbb{R}^{p\mathbb{Z}^d})$ and $V \in \mathscr{B}(\mathbb{R})$ such that

$$0 < \int_V e^{-x}\mathrm{d}x < \infty.$$

Furthermore, let $\Pi = \sum_{i\in\mathbb{N}} \delta_{(T_i^{(p)}, X_i+M_i^{(p)}, F_i^{(p)})}$ the Poisson point process on $p\mathbb{Z}^d \times \mathbb{R} \times \mathbb{R}^{p\mathbb{Z}^d}$ with the notations from Remark 7. Then, we have

$$\tilde{Q}^{(p)}(A) = \mathbb{P}(\Pi(\{0\} \times V \times A) = 1 \mid \Pi(\{0\} \times V \times \mathbb{R}^{p\mathbb{Z}^d}) = 1), \tag{3}$$

since the intensity measure of Π is a product measure.

Since we may assume that the points $(T_i^{(p)}, X_i + M_i^{(p)}, F_i^{(p)})$ are numbered such that the sequence $(X_i)_{i\in\mathbb{N}}$ is decreasing (cf. Section 5), we get

$$\begin{aligned}
&\mathbb{P}\left(\Pi(\{0\} \times V \times A) = 1 \mid \Pi(\{0\} \times V \times \mathbb{R}^{p\mathbb{Z}^d}) = 1\right)\\
&\ = \sum_{i\in\mathbb{N}} \mathbb{P}\left(T_i^{(p)} = 0,\ X_i + M_i^{(p)} \in V \,\middle|\, \#\left\{i : \left(T_i^{(p)}, X_i + M_i^{(p)}\right) \in \{0\} \times V\right\} = 1\right)\\
&\quad \cdot \mathbb{P}\left(F_i^{(p)} \in A \,\middle|\, T_i^{(p)} = 0,\ X_i + M_i^{(p)} \in V,\right.\\
&\qquad\quad \left.\#\left\{i : \left(T_i^{(p)}, X_i + M_i^{(p)}\right) \in \{0\} \times V\right\} = 1\right)
\end{aligned} \tag{4}$$

with

$$\mathbb{P}\left(F_i^{(p)} \in A \mid T_i^{(p)} = 0,\ X_i + M_i^{(p)} \in V, \#\{i:\ (T_i^{(p)}, X_i + M_i^{(p)}) \in \{0\} \times V\} = 1\right)$$

$$= \mathbb{P}\left(F_i^{(p)} \in A \mid T_i^{(p)} = 0,\ X_i \in V,\ (T_j^{(p)}, X_j + M_j^{(p)}) \notin \{0\} \times V\ \forall j \neq i\right)$$

$$= \mathbb{P}\left(F_i^{(p)} \in A \mid T_i^{(p)} = 0\right),$$

where we use the fact that W_i is independent of X_i, X_j and W_j for all $j \neq i$.

Employing Eqs. 3, 4, and

$$\sum_{i \in \mathbb{N}} \mathbb{P}\left(T_i^{(p)} = 0,\ X_i + M_i^{(p)} \in V \mid \Pi(\{0\} \times V \times \mathbb{R}^{p\mathbb{Z}^d}) = 1\right) = 1$$

we get

$$\tilde{Q}^{(p)}(A) = \mathbb{P}\left(F_i^{(p)} \in A \mid T_i = 0\right) = \mathbb{P}\left(W(\cdot) - \frac{\sigma^2(\cdot)}{2} \in A \mid T^{(p)} = 0\right)$$

for all $A \in \mathscr{B}(\mathbb{R}^{p\mathbb{Z}^d})$. □

Remark 9 Let Π be defined as in the proof of Theorem 8. Considering the intensity $\lambda^{(p)} p^d$ of the restriction of Π on the set $\{0\} \times [0, \infty) \times C(\mathbb{R}^d)$ we get the equality $\lambda^{(p)} p^d = \mathbb{P}(T^{(p)} = 0)$.

Using only the assumptions of Theorem 6, $\tilde{Q}$ can be described as the law of F_i conditional on $X_i + M_i$ and T_i. Let $\Pi = \sum_{i \in \mathbb{N}} \delta_{(T_i, X_i + M_i, F_i)}$ and $E \in \mathscr{B}(\mathbb{R}^d \times \mathbb{R})$ such that $\int_E e^{-x}(\mathrm{d}t \times \mathrm{d}x) \in (0, \infty)$. Furthermore, let $N = \Pi(E \times \mathscr{C}(\mathbb{R}^d))$ and $i_1, \ldots, i_N$ such that $(T_{i_k}, X_{i_k} + M_{i_k}) \in E$ for $k = 1, \ldots, N$. By $G_1, \ldots, G_N$ we denote a random permutation of $F_{i_1}, \ldots, F_{i_N}$.

Theorem 10 *Conditional on $N = n$, the processes $G_1, \ldots, G_n$ are i.i.d. with law $\tilde{Q}$.*

Proof We have to proof that all finite dimensional margins of $G_1, \ldots, G_n$ are products of one dimensional margins with law $\tilde{Q}$. By decomposing the sets of $\mathscr{C}$ and changing numbering of indices it suffices to proof that $\mathbb{P}(G_1 \in A_1, \ldots, G_{n_1} \in$

$A_1, G_{n_1+1} \in A_2, \ldots, G_{n_1+n_2+\ldots+n_l} \in A_l \mid N = n)$ equals $\prod_{i=1}^l \tilde{Q}(A_i)^{n_i}$ for pairwise disjoint sets $A_1, \ldots, A_l \in \mathscr{C}$, $n_1, \ldots, n_l \in \mathbb{N}$ with $n_1 + \ldots + n_l \leq n$. Let $m = n_1 + \ldots + n_l$ and $A = \bigcup_{i=1}^l A_i$. Then, we have

$$
\begin{aligned}
&\mathbb{P}\left(G_1 \in A_1, \ldots, G_{n_1} \in A_1, G_{n_1+1} \in A_2, \ldots, G_m \in A_l \mid N = n\right)\\
&= \sum_{\substack{k_1 \geq n_1, \ldots, k_l \geq n_l \\ k_1 + \ldots + k_l \leq n}} \mathbb{P}\left(G_1 \in A_1, \ldots, G_m \in A_l \,\middle|\, \bigcap_{j=1}^l \Pi(E \times A_j) = k_j,\ N = n\right)\\
&\quad \cdot \mathbb{P}\left(\Pi(E \times A_j) = k_j,\ j = 1, \ldots, l \mid N = n\right)\\
&= \sum_{\substack{k_1 \geq n_1, \ldots, k_l \geq n_l \\ k_1 + \ldots + k_l \leq n}} \frac{k_1}{n} \cdots \frac{k_1 - n_1 + 1}{n - n - 1 + 1} \frac{k_2}{n - n_1} \cdots \frac{k_l - n_l}{n - m + 1}\\
&\quad \cdot \binom{n}{k_1, \ldots,\ k_l,\ n - k_1 - \ldots - k_l} \cdot \tilde{Q}(A_1)^{k_1} \cdots \tilde{Q}(A_l)^{k_l}\\
&\quad \cdot (1 - \tilde{Q}(A))^{n - k_1 - \ldots - k_l}\\
&= \prod_{i=1}^l \tilde{Q}(A_i)^{n_i}.
\end{aligned}
$$

□

5 Finite approximations

Let $Y_1, Y_2, \ldots$ be independent exponentially distributed random variables with parameter $\lambda > 0$ and define $R_n = \sum_{i=1}^n Y_i$ for $n \in \mathbb{N}$. Then, $\sum_{i \in \mathbb{N}} \delta_{R_i}$ is a Poisson point process on $(0, \infty)$ with intensity λ. Applying the mapping theorem (Kingman 1993) we get that $\sum_{i \in \mathbb{N}} \delta_{-\log R_i}$ is a Poisson point process on $\mathbb{R}$ with intensity measure $\lambda \exp(-x)\mathrm{d}x$ and the sequence $(X_i)_{i \in \mathbb{N}}$ with $X_i = -\log R_i$ is monotonically decreasing.

For simplicity we will only consider approximations of Z on a symmetric cuboid $[-b, b]$, $b \in \mathbb{R}^d$, based on the definition of Z and the representations Z_1, Z_2, Z_3, and Z_4, respectively.

1. For the Poisson point process $\Pi = \sum_{i \in \mathbb{N}} \delta_{X_i}$ with intensity measure $\exp(-x)\mathrm{d}x$ and independent copies $\{W_i(t),\ t \in [-b, b]\}$ of a Gaussian process with stationary increments, let

$$Z^{(k)}(t) = \max_{i=1,\ldots,k} \left(X_i + W_i(t) - \sigma^2(t)/2\right), \quad t \in [-b, b],\ k \in \mathbb{N}.$$

2. Let $\{X_i^{(j)}\}_{i \in \mathbb{N}}$, $j = 1, \ldots, n$, be decreasing sequences of points of the Poisson point processes $\sum_{i \in \mathbb{N}} \delta_{X_i^{(j)}}$ with intensity measure $n^{-1} \exp(-x)\mathrm{d}x$ and $\{W_i^{(j)}(t),$

$t \in [-b - h_j,\ b - h_j]\}$ be independent copies of Gaussian processes. Then, for $k \in \mathbb{N}$, let

$$Z_1^{(k)}(t) = \max_{j=1,\dots,n} \max_{i=1,\dots,k} \left(X_i^{(j)} + W_i^{(j)}(t - h_j) - \frac{\sigma^2(t - h_j)}{2} \right), \quad t \in [-b, b].$$

3. Let $\sum_{i \in \mathbb{N}} \delta_{(X_i, H_i)}$ be a Poisson point process on $I \times \mathbb{R}$ with intensity measure $|I|^{-1} \mathrm{d}h \times \exp(-x)\mathrm{d}x$. For each $i \in \mathbb{N}$, let $\{W_i(t),\ t \in [-b - I_{\max},\ b - I_{\min}]\}$ be independent copies of a Gaussian process where $I_{\min}$ and $I_{\max}$ are the lower and upper end point of I, and let

$$Z_2^{(k)}(t) = \max_{i=1,\dots,k} \left(X_i + W_i(t - H_i) + \sigma^2(t - H_i)/2 \right), \quad t \in [-b, b],\ k \in \mathbb{N}.$$

4. For $j_{\min} \leq j \leq j_{\max} \in \mathbb{Z}^d$, let $\{X_i^{(j)}\}_{i \in \mathbb{N}}$ be descending sequences of points of the Poisson point processes $\sum_{i \in \mathbb{N}} \delta_{X_i^{(j)}}$ with intensity measure $m^{-d} \exp(-x)\mathrm{d}x$. We assume $pj_{\min} < a < b < pj_{\max}$. Furthermore, we have independent copies $\{W_i^{(j)},\ t \in [-b - pj,\ b - pj]\}$ of Gaussian process and define $T_i^{(j)} = \inf(\operatorname{argsup}(W_i^{(j)}(t) - \frac{\sigma^2(t)}{2}))$. For $k \in \mathbb{N},\ t \in [-b, b]$, let

$$Z_3^{(k)}(t) = \max_{j=j_{\min},\dots,j_{\max}} \max_{\substack{i=1,\dots,k \\ T_i^{(j)} \in (-\frac{m}{2}p, \frac{m}{2}p]^d}} \left(X_i^{(j)} + W_i^{(j)}(t - pj) - \frac{\sigma^2(t - pj)}{2} \right).$$

5. Let I be a finite interval in $\mathbb{R}^d$ with $[-b, b] \subset I$. Let (U_i, S_i) be descending in U_i such that $\sum_{i \in \mathbb{N}} \delta_{(U_i, S_i)}$ is a Poisson point process on $\mathbb{R} \times I$ with intensity measure $\lambda^* \exp(-u)\mathrm{d}u \times \mathrm{d}s$, i.e. $U_1 \geq U_2 \geq U_3 \geq \dots$. For each $i \in \mathbb{N}$, let $\tilde{F}_i$ be an independent sample path with law $\tilde{Q}$.
Then, for $k \in \mathbb{N},\ t \in [-b, b]$, let

$$Z_4^{(k)}(t) = \max_{i=1,\dots,k} \left(U_i + \tilde{F}_i(t - S_i) \right).$$

This construction is illustrated by Fig. 2.

For simulating $\tilde{F}_i \sim_{i.i.d.} \tilde{Q}$ or the discretized version $\tilde{F}_i^{(p)} \sim_{i.i.d.} \tilde{Q}^{(p)}$ we can either use Theorem 8 or 10. In the first case we simulate independent copies $W_j(\cdot)$ of $W(\cdot)$ and reject all those processes with $T_j^{(p)} \neq 0$. In the second case we simulate a Brown–Resnick process (e.g. using the standard approximiation method) and use all those processes F_j with $(X_j + M_j, T_i) \in E$ in a random order. We have to choose E carefully such that we have to simulate as few processes $W_j(\cdot)$ as possible to get a realisation of $\tilde{F}_i$.

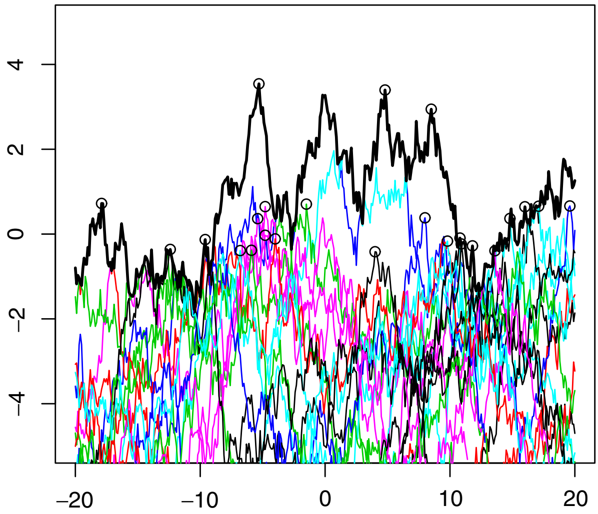

Fig. 2 Construction of one sample path of the original Brown–Resnick process based on the representation $Z(\cdot) = Z_4(\cdot)$. The *circles* mark the points (U_i, S_i). The resulting process is displayed by the *bold line*.

The constants λ^* or $\lambda^{(p)} p^d$ can be estimated by counting $\#\{i \in \mathbb{N} : \ X_i > 0,\ T_i \in [0,1]^d\}$ or $\#\{i \in \mathbb{N} : \ X_i > 0,\ T_i^{(p)} = 0\}$ (i.e. estimating $\mathbb{P}(T^{(p)} = 0)$), respectively.

6 Error estimates

In this section we assume that $\{W(t),\ t \in \mathbb{R}\}$ is a one-dimensional standard Brownian motion. Furthermore, we consider a symmetric interval $[-b, b]$ on which the simulation is performed.

In order to estimate the error for the approximation of $Z(\cdot)$ by $Z^{(k)}(\cdot)$, we define the random variable $C_k = \inf_{t\in[-b,b]}(Z^{(k)}(t) + \sigma^2(t)/2)$. Then, we get the following result.

Proposition 11 (Oesting 2010, Proposition 3.8) *Let $x < c < 0$. Then,*

$$\mathbb{P}\left(Z^{(k)}(t) \neq Z(t) \text{ for some } t \in [-b, b] \mid X_k \leq x, C_k > c\right)$$
$$\leq 4e^{-c} \frac{b}{c-x} \exp\left(\frac{b}{2}\right) \left(1 - \Phi\left(\frac{c-x-b}{\sqrt{b}}\right)\right). \tag{5}$$

Furthermore, we have

$$\mathbb{P}(C_k \leq c) \leq 1 - \left(1 - \exp\left(-\exp\left(-\frac{c}{2}\right)\right)\right) \cdot \left(2\Phi\left(-\frac{c}{2\sqrt{b}}\right) - 1\right)^2 \tag{6}$$

and

$$\mathbb{P}(X_k > x) \leq \frac{\exp(-(k-1)x)}{(k-1)!}(1 - \exp(-\exp(-x))). \tag{7}$$

The unconditional error probability $\mathbb{P}(Z^{(k)}(t) \neq Z(t)$ *for some* $t \in [-b, b])$ *can be bounded by the sum of the rhs of Eqs. 5, 6 and 7.*

If the right-hand sides of Eqs. 5, 6 and 7 tend to 0 for $k \to \infty$, e.g. for $c = c(k) = -\log\log(\log(k)/2)$ and $x = x(k) = -\log(k)/2$, we get

$$\lim_{k\to\infty} \mathbb{P}(Z^{(k)}(t) \neq Z(t) \text{ for some } t \in [-b, b]) = 0.$$

Similar results hold for $Z_1^{(k)}(\cdot)$ and $Z_2^{(k)}(\cdot)$, cf. Oesting (2010). In particular, these bounds have the same asymptotic behaviour.

To give bounds for $Z_4(\cdot)$, we consider the standard Brownian motion restricted to the lattice $p\mathbb{Z}$ for some $p > 0$. Here, we also have to adjust the Poisson point process $\sum_{i\in\mathbb{N}} \delta_{(U_i,S_i)}$ to the lattice and to restrict its second component to some finite interval $[-v, v]$. Let $\sum_{i\in\mathbb{N}} \delta_{(Y_i^{(p)},R_i)}$ a Poisson point process on $\mathbb{R} \times [-v, v]$ with intensity measure $\lambda^{(p)} p \exp(-y)\mathrm{d}y \times \sum_{k\in\mathbb{Z}\cap[-v/p,v/p]} \delta_{pk}(\mathrm{d}s)$. Then, we get an approximation of Z_4 by

$$Z_4^{(k,v,p)}(t) = \max_{i=1,\dots,k} \left(Y_i + \tilde{F}_i(t - R_i)\right)$$

for $t \in [-b, b] \cap p\mathbb{Z}$, where $\tilde{F}_i \sim_{i.i.d.} \tilde{Q}^{(p)}$. For the following result we use that $\tilde{Q}^{(p)}$ is the law of $W(t) - \frac{|t|}{2} \mid T^{(p)} = 0$, where W is a standard Brownian motion and $T^{(p)} = \inf\left(\operatorname{argsup}_{t\in p\mathbb{Z}} (W(t) - |t|/2)\right)$.

Proposition 12 (Oesting 2010, Proposition 3.17) *Let* $b, v \in p\mathbb{N}$ *with* $v - b \geq 16$ *and* $b \geq 1$. *Furthermore, let* $C_k = \min_{t\in[-b,b]\cap p\mathbb{Z}} Z_4^{(k,v,p)}(t)$ *and* $c < 0$. *Then, we have*

$$\begin{aligned}&\mathbb{P}\left(Z_4^{(k,v,p)}(t) \neq Z_4(t) \textit{ for some } t \in [-b, b] \cap p\mathbb{Z} \mid C_k > c,\ Y_k^{(p)} \leq c\right)\\ &\leq \frac{64\lambda^{(p)} e^{-c}}{(1 - \exp(-p/2))^2} \left(\exp\left(-\frac{\sqrt{v-b}}{2}\right) + 7\sqrt{b^3} \exp\left(-\frac{v-b}{48b}\right)\right).\end{aligned} \tag{8}$$

Furthermore, it holds that

$$
\begin{aligned}
\mathbb{P}(C_k \leq c) \leq 1 - \Big(1 - \exp\Big(-\lambda^{(p)}(2b+p)e^{-c/2}\Big)\Big) \\
\cdot \left(1 - \Big(2(o-b)\lambda^{(p)}\Big)^k \frac{\exp\left(-\frac{(k-1)c}{2}\right)}{(k-1)!}\Big(1 - \exp\Big(-e^{-(v-b)\lambda^{(p)}c}\Big)\Big)\right) \\
\cdot \left(1 - \frac{4}{1-e^{-p/2}}\left(1 - \Phi\left(-\frac{c}{\sqrt{8b}} - \sqrt{\frac{b}{2}}\right)\right.\right. \\
\left.\left. - \exp(-c/2)\left(1 - \Phi\left(-\frac{c}{\sqrt{8b}} + \sqrt{\frac{b}{2}}\right)\right)\right)^2\right)
\end{aligned} \tag{9}
$$

and

$$
\begin{aligned}
\mathbb{P}\left(Y_k^{(p)} > c\right) \leq \Big((2v+p)\lambda^{(p)}\Big)^k \\
\cdot \frac{\exp(-(k-1)c)}{(k-1)!}\Big(1 - \exp\Big(-e^{-(2v+p)\lambda^{(p)}c}\Big)\Big).
\end{aligned} \tag{10}
$$

The unconditional error probability $\mathbb{P}(Z_4^{(k,v,p)}(t) \neq Z_4(t)$ for some $t \in [-b, b])$ can be bounded by the sum of the rhs of Eqs. 8, 9 and 10.

If we choose $v = v(k) = k^{1/4}$ and $c = c(k) = -\log(k)/4$ for example, again all the error bounds given in Proposition 12 tend to 0 as $k \to \infty$.

Here, the unconditional bound is given for fixed $k \in \mathbb{N}$. On the other hand, we can also use a stopping rule similar to Schlather (2002) and consider $Z_4^{(k)}$ with $k \in \mathbb{N}$ such that $C_k > Y_k^{(p)}$. Then, the error can be assessed by the bound for the first probability which is independent of k.

Note, that the upper bounds given in Proposition 12 can be made explicite by employing $\lambda^{(p)} = \mathbb{P}(T^{(p)} = 0)/p$ (cf. Remark 9) and using further bounds like $(1 - \exp(-p/2))^2/4 \leq \mathbb{P}(T^{(p)} = 0) \leq 1$. Similar bounds can be given for $Z_3^{(k)}$, cf. Oesting (2010).

7 Simulation study

In order to compare the different simulation techniques described in Section 5 we perform a simulation study on $\mathbb{R}^1$ using the software environment `R` (Ihaka and

Gentleman 1996; R Development Core Team 2009). We consider symmetric intervals $[-b, b]$ with $b \in \{1, 2, 5, 10, 20, 30, 50\}$ and the variogram $\gamma(h) = 2|h|^\alpha$ for $\alpha \in \{0.2, 0.4, 0.6, \ldots, 1.8\}$. This means that, for all Brown–Resnick processes, condition (1) holds. We always consider the process on a grid of length 0.1.

In order to get a fair criterion for the comparison of the different methods we have fixed the number q of simulated sample paths of $W(\cdot)$ on $[-b, b]$ per realization of the Brown–Resnick process, $q = 100$, 500 and 2,500. Simulation techniques for W are given in Lantuéjoul (2002) and Schlather (2009). In order to approximate Z_3 and Z_4 the pathes $W(\cdot)$ have to be computed on an interval larger than $[-b, b]$. Here, we assume that the computing time depends linearly on the length of this interval and modify the number of simulated sample paths to have an approximately equal computing time for all the approximations. We repeat any simulation of a Brown–Resnick process $N = 5{,}000$ times and call this a run.

For ease, we will call the approximation of Z_i "method i" ($i = 1, 2, 3, 4$) and the approximation of Z "method 0".

Applying method 1, we have to choose $h_1, \ldots, h_n$ depending on b. It seems to be reasonable to distribute $h_1, \ldots, h_n$ equally on $[-b, b]$. Furthermore, the distance $\Delta h = h_2 - h_1$ should neither be too large—because we want to cover the interval with good approximations—nor too small—in order to get a method distinct from method 2. Here, depending on b, we choose some $n \in \{5, 10, 20, 50\}$ such that $0.5 \leq \Delta h \leq 2$.

In order to approximate $Z_2(\cdot)$, let Q be the uniform distribution on $[-b, b]$. When approximating $Z_3(\cdot)$, we always set p as the mesh width, i.e. $p = 0.1$, and set $j_{\max} = 200$ for $b = 1, 2$, $j_{\max} = 250$ for $b = 5$ and $j_{\max} = 10b + 300$ otherwise. Note that for the choice of $j_{\max}$ (and also for the choice of v in method 4) not its absolute value, but the difference $pj_{\max} - b$ is important. In practice, a difference of 30 (or larger) provides very good results. On the other hand, one should be aware of the fact, that increasing $j_{\max}$ is quite expensive in terms of computing times if k is fixed. We also varied the intensity parameter m and got best results for $m = 31$.

In case of method 4 we set $v = 20$ for $b = 1, 2$, $v = 25$ for $b = 5$ and $v = b + 30$ otherwise. Here, the choice of the set E is crucial. A large domain of E requires the simulation of low values of X_i, involving high computational costs. A very small domain, however, leads to a high rejection rate. We choose E in the following way. Let $\tilde{\Pi} = \sum_{i \in \mathbb{N}} \delta_{(X_i, T_i, X_i + M_i)}$ be a Poisson point process on $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ with intensity measure $\tilde{\Lambda}$. From Theorem 6 it is known that $\tilde{\Lambda}(\mathbb{R} \times E) = \lambda^* \int_E e^{-y} \mathrm{d}t \times \mathrm{d}y$. We compare this to $\tilde{\Lambda}([x_0, \infty) \times E)$ for some fixed $x_0 \in \mathbb{R}$. The latter one can be easily estimated by simulation. Then, the probability of drawing $\tilde{F}_i$ incorrectly when restricting our simulation to processes with $X_j > x_0$ conditional on $\tilde{\Pi}(\mathbb{R} \times E) > 0$ is bounded by $1 - \exp(-\tilde{\Lambda}((-\infty, x_0) \times E))$. For our simulation study we choose $x_0 = -2$ and approximate the area of highest intensity with cubes.

As already mentioned before, for methods 3 and 4, the (location of the) maximum of the Gaussian process is needed. To this end, we simulate the Gaussian process on a larger interval, which is at least of length $pj_{\max} + b$ or $v + b$, respectively, and take the maximum of the process restricted to this area which implies additional errors. Note that we do not get any additional error by discretization as the equivalent representations $Z_3(\cdot)$ and $Z_4(\cdot)$ also hold for Brown–Resnick processes restricted to a lattice (cf. Remark 7).

As a measure of approximation quality we took the largest distance between the empirical cumulative distribution function of the approximated process at the interval bounds and the standard Gumbel distribution function. This is motivated by the fact that we expect the largest deviations from the original process at the interval bounds. Both bounds are taken into account in order to get a lower volatility of the results: For independent realisations $Z_{i,1}^{(k)}(\cdot), \ldots, Z_{i,N}^{(k)}(\cdot)$ let $Z_{i,(1)}^{(k)}(t), \ldots, Z_{i,(N)}^{(k)}(t)$ be the order statistic at location $t \in p\mathbb{Z}$. Then, we define the deviation of approximation as

$$\varepsilon = \frac{1}{2} \max_{j=1,\ldots,N} \left| \exp\left(-\exp\left(-Z_{i,(j)}^{(k)}(-b)\right)\right) - \frac{j}{N} \right| + \frac{1}{2} \max_{j=1,\ldots,N} \left| \exp\left(-\exp\left(-Z_{i,(j)}^{(k)}(b)\right)\right) - \frac{j}{N} \right|.$$

We perform all the simulations up to 50 times. After each run of all methods we calculate the p-values for pairwise t-tests between the different methods assuming that ε is normally distributed. We stop simulating a method whenever $p < 0.005$. Figure 3 depicts the methods for each pair (b, α) that have not been rejected after 50 repetitions.

In general, methods 0, 1 and 2 perform best, if α or b is small. If both are large, method 4 is the best one. The area where method 4 performs best increases for q growing. For large q methods 0 and 2 have the same behaviour; if q is small, there is a sharper distinction between these methods. Method 0 provides better results for

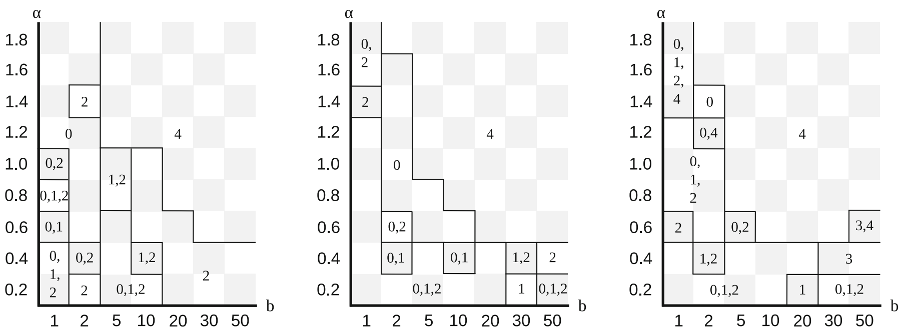

Fig. 3 Methods providing best results depending on the interval bound b and variogram parameter α using $q = 100$ (*left*), 500 (*middle*), and 2,500 (*right*) simulated sample paths of $W(\cdot)$ on $[-b, b]$ per realization of the Brown–Resnick process.

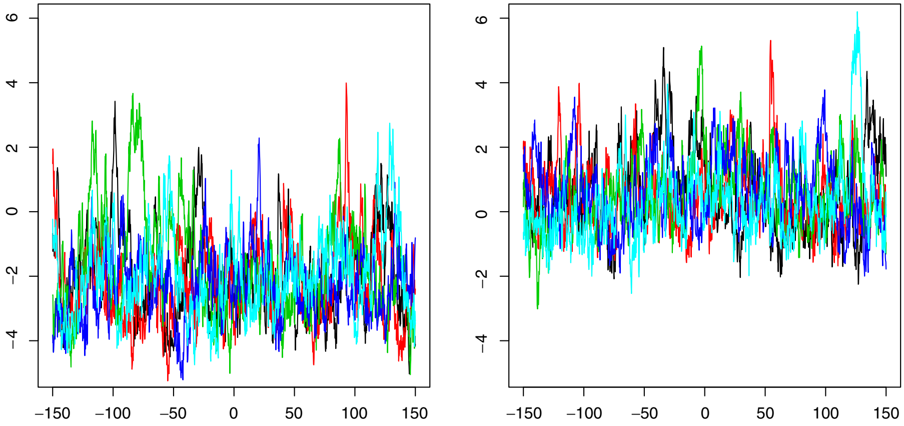

Fig. 4 Finite approximations of the original Brown–Resnick process; five realizations of $Z_2^{(1000)}(\cdot)$ (*left*) and $Z_4^{(1000)}(\cdot)$ (*right*).

small b, method 2 for small α. Method 3 only works well if q gets large. Then, we get best results for b large.

The typical behaviour for large b, α and q is shown in Figs. 1 and 4 for the standard Brownian motion. The development of the deviation ε for growing q and different b and α is shown in Table 1. More generally, there are the following recommendations concerning the choice of methods in practice: If the variogram value evaluated at the diameter of the simulated area is low, then use the original definition; simulation by random shifting is also appropriate if an unprecise simulation is sufficient; if the variogram tends to infinity and the value evaluated at the diameter of the simulated window is high, then the mixed moving maxima representation is best.

Table 1 The mean deviation of approximation is shown for different (α, b, q)

	$\alpha = 0.4, b = 30$			$\alpha = 1, b = 10$			$\alpha = 1.6, b = 2$		
q	100	500	2,500	100	500	2,500	100	500	2,500
ε_0	0.148	0.066	0.027	0.662	0.514	0.367	0.085	0.030	0.014
ε_1	0.153	0.064	0.026	0.429	0.345	0.281	0.137	0.101	0.065
ε_2	0.135	0.063	0.030	0.423	0.339	0.280	0.129	0.084	0.046
ε_3	0.816	0.472	0.023	0.833	0.493	0.026	0.848	0.519	0.025
ε_4	0.379	0.099	0.066	0.514	0.076	0.014	0.357	0.045	0.012

By ε_i we denote the deviation of approximation by method i, $i = 0, \ldots, 4$

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